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Diagonalization in Jordan Pairs

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INTRODUCTION

As every student of linear algebra knows, a rectangular matrix over a division ring can be diagonalized by elementary row and column operations. Similarly, there are normal forms for alternating and hermitian matrices which can be achieved by simultaneous row and column operations. Since matrices of this kind form the main examples of Jordan pairs, it is natural to ask whether similar results hold in the Jordan setting. We show that this is in fact the case, study the obstruction to diagonalizability, the defect, and also prove that nondegenerate Jordan pairs admit a rank function sharing many properties with classical matrix rank.

Let $V=(V^+, V^-)$ be a Jordan pair and $S=\{e_1, ..., e_r\}$ a set of orthogonal idempotents. The *S-diagonal elements* are those in $\sum_{i=1}^r V_2(e_i)$. Now suppose that V is nondegenerate and satisfies dcc on principal inner ideals. An element $x \in V^\sigma$ ($\sigma = \pm$) is called *diagonalizable* if x is S-diagonal for some S consisting of division idempotents. To see that, for matrices, this is the same as the usual notion of diagonalizability, suppose that V has in addition acc on principal inner ideals. Then V contains a *frame*, that is, a finite set F of orthogonal division idempotents such that $V_0(F)=0$. For rectangular or hermitian matrices over division rings, F can be taken to consist of the diagonal matrix units ($e_i^\pm = E_{ii}$) whereas for alternating matrices over a field, $e_i^\pm = E_{2i-1,2i} - E_{2i,2i-1}$. The Jordan analogues of elementary row and column operations are the inner automorphisms $\beta(V_1^+(e), V_2^-(e))$ and $\beta(V_2^+(e), V_1^-(e))$ (where $e \in F$) which generate the group of F-elementary automorphisms of V [6]. Any set of orthogonal division idempotents can be transformed (up to association) into F by an F-elementary automorphism [6, Th. 2]. It follows that x is diagonalizable if and only if $\varphi_+(x)$ is F-diagonal for some elementary automorphism φ of V. This shows the equivalence with the usual definition.

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A symmetric matrix x over a field is diagonalizable if and only if it is not alternating; i.e., ${}^t vxv \neq 0$ for some column vector v (cf. [1, Th. 6]). It is an easy exercise to show that this is equivalent to the condition $yxy \neq 0$, for some symmetric rank one matrix y. In the Jordan setting, rank one matrices correspond to *simple elements*, that is, elements generating simple (=minimal nontrivial) inner ideals; in fact, the simple elements are precisely the elements of rank one with respect to the rank function discussed later. Now our diagonalization theorem reads as follows (Theorem 1): *An element $x \neq 0$ of a simple Jordan pair with dcc on principal inner ideals is diagonalizable if and only if $Q_y x \neq 0$ for some simple element y.* The well-known normal forms of matrices alluded to at the beginning are all contained in this theorem.

By Theorem 1, the obstruction to diagonalizability lies in the *defect* Def V, defined as the set of all x with $Q_y x = 0$ for all simple y. Defectiveness is a characteristic 2 phenomenon due to the quadratic nature of the Jordan product: Even though every z is a sum of simple elements (i.e., V equals its socle) and $Q_z x = 0$ for all z implies $x = 0$, there may be nonzero defective elements. It turns out (Theorem 2) that Def V is an outer ideal which is itself nondegenerate with dcc on principal inner ideals and has defect zero.

As an illustration, let V be the Jordan pair defined by a quadratic form q (with associated bilinear form b) on a vector space $X = V^+ = V^-$ with Jordan product $Q_x y = b(x, y)\, x - q(x)\, y$, and assume q is nondegenerate ($q(x) = b(x, X) = 0$ implies $x = 0$). If u, u' is a hyperbolic pair spanning the hyperbolic plane H then $e_1 = (u, u')$, $e_2 = (u', u)$ is a frame of V. For $v \in H^\perp = V_1^\pm(e_1)$, the Bergmann operators $B(u, -v)$ and $B(v, u)$ agree, by a simple computation, with the Eichler transformation $\sum_{u,v}$ [2, 5.2.9], and the defect of V is just the radical of b. Thus any x not in the radical of b can be mapped into H by a product of Eichler transformations.

Let V be a nondegenerate Jordan pair. The *rank* of an element x, denoted $\mathrm{rk}(x)$, is the supremum of the lengths of chains of principal inner ideals determined by elements in the inner ideal generated by x. Just as for rings of linear transformations [11, Th. 2.1.25], the socle of V consists precisely of the finite rank elements. The rank function satisfies various properties familiar from matrix rank, notably the triangle inequality, and does not increase under structural transformations (Theorem 3). There is also a Jordan analogue of the Frobenius inequality involving Peirce gradings (Theorem 4). The proofs of these facts are complicated by the defect phenomenon: In contrast to ring theory [11, Lemma 2.1.17], it is not true that every x of rank r is the sum of r rank one elements. In fact, this is equivalent to diagonalizability, and a defective x of rank r can be represented as a sum of $r+1$ but not of r rank one elements.

Notation and terminology follow [5]. All Jordan pairs are modules over an arbitrary commutative ring k of scalars. We will also frequently use the results of [6, 7], in particular the fact that a nondegenerate Jordan pair satisfies the dcc for principal inner ideals (principal dcc) if and only if it is equal to its socle, and that such Jordan pairs are (von Neumann) regular.

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1. Diagonalization and Defect

Let V be a nondegenerate Jordan pair with principal dcc. Since we can replace V by $V^{\mathrm{op}} = (V^-, V^+)$, it will usually be sufficient to consider elements of V^+. The following weak diagonal form (cf. [1, Lemma 7]) is always possible.

Lemma 1. *For every $x \in V^+$ there exists a set $S = \{e_1, ..., e_r\}$ of orthogonal division idempotents such that*

$$x = \sum_{i=1}^{s} x_{2i-1,2i} + \sum_{j=2s+1}^{2s+t} x_{jj}$$

(where x_{ij} is in the Peirce space V^+_{ij} relative to S) with $x_{2i-1,2i}$ invertible in $V_{2i-1,2i-1} \oplus V_{2i-1,2i} \oplus V_{2i,2i}(i = 1, ..., s)$, and x_{jj} invertible in $V_{jj}(j = 2s+1, ..., 2s+t = r)$.

Proof. By [7, Th. 1], $x = c_+$ can be completed to an idempotent c, and $V_2(c)$ has finite capacity. By [6, Th. 1], there exists a frame $\{e_1, ..., e_r\}$ of $V_2(c)$ with the required property.

Nondiagonalizability is a characteristic 2 phenomenon:

Proposition 1. *If V is nondegenerate with principal dcc and $2V = V$ then every element is diagonalizable.*

Proof. By Lemma 1, it suffices to show that an invertible $x_{12} \in V^+_{12}$ can be diagonalized in $W = V_2(e_1 + e_2) = V_{11} \oplus V_{12} \oplus V_{22}$, where e_1, e_2 are orthogonal division idempotents. First note that V may be regarded as a Jordan pair over $\mathbf{Z}[\frac{1}{2}]$. Indeed, multiplication by 2 is injective: If $2x = 0$ then $Q_{2x} V^- = 4Q_x V^- = Q_x 4V^- = Q_x V^- = 0$ which implies $x = 0$ by nondegeneracy, and one shows easily that the Jordan product $Q_x y$ is quadratic in x and linear in y over $\mathbf{Z}[\frac{1}{2}]$. Now let $y_{12} \in V^-_{12}$ be the inverse of x_{12}. Then $B(e_1^+, -\frac{1}{2} y_{12}) x_{12} = x_{12} + \frac{1}{2}\{e_1^+, y_{12}, x_{12}\} = x_{12} + e_1^+$ by the Peirce relations and since $\{x_{12}, y_{12}, z\} = 2z$ for all $z \in W^+$. Hence by Lemma 1 of [6] or a direct calculation as above,

$$B(x_{12}, e_1^-)\, B(e_1^+, -\tfrac{1}{2} y_{12})\, x_{12} = e_1^+ - Q(x_{12})\, e_1^- \in V^+_{11} \oplus V^+_{22},$$

and x_{12} is diagonal with respect to $S=\{\varphi(e_1), \varphi(e_2)\}$, where

$$\varphi=\beta(e_1^+, \tfrac{1}{2}y_{12})\,\beta(-x_{12}, e_1^-).$$

LEMMA 2. *Let V be an arbitrary Jordan pair, let $S=\{e_1, e_2, e_3\}$ be a set of three orthogonal connected idempotents with Peirce spaces V_{ij}, and let $x=x_{12}+x_{33}$ where $x_{12}\in V_{12}^+$ is invertible in $V_{11}\oplus V_{12}\oplus V_{22}$, and $x_{33}\in V_{33}^+$ is invertible in V_{33}. Then an S-elementary transform of x is S-diagonal; more precisely,*

$$\varphi_+(x)=x_{11}-x_{22}+x_{33}$$

where x_{ii} is invertible in V_{ii} and

$$\varphi=\beta(-x_{12}, x_{22}^{-1})\,\beta(-x_{23}, x_{33}^{-1})\,\beta(x_{33}, y_{13})$$

for suitable x_{23}, y_{13} in the corresponding Peirce spaces.

Proof. Choose y_{13} invertible in $V_{11}\oplus V_{13}\oplus V_{33}$, and let $x_{23}=\{x_{12}, y_{13}, x_{33}\}$. Then $x_{22}=Q(x_{23})\,x_{33}^{-1}$ is invertible in V_{22}: Indeed, from the identity JP20 [5, p. 19] and the Peirce relations it follows by a simple computation that $x_{22}=Q(x_{12})\,Q(y_{13})\,x_{33}$. By relative invertibility of y_{13} we have $Q(y_{13})\,x_{33}$ invertible in V_{11}, and by an analogous argument x_{22} is invertible in V_{22} and $x_{11}=Q(x_{12})\,x_{22}^{-1}$ is invertible in V_{11}. Now compute, using the Peirce relations:

$$\begin{aligned}
B(x_{33}, y_{13})(x_{12}+x_{33}) &= x_{12}+x_{33}-\{x_{33}, y_{13}, x_{12}+x_{33}\}=x_{12}+x_{33}-x_{23},\\
B(-x_{23}, x_{33}^{-1})&(x_{12}-x_{23}+x_{33})\\
&= x_{12}-x_{23}+x_{33}+\{x_{23}, x_{33}^{-1}, x_{33}+x_{12}-x_{23}\}+Q(x_{23})\,Q(x_{33}^{-1})\,x_{33}\\
&= x_{12}-x_{23}+x_{33}+x_{23}-2Q(x_{23})\,x_{33}^{-1}+Q(x_{23})\,x_{33}^{-1}\\
&= x_{12}-x_{22}+x_{33},
\end{aligned}$$

and finally

$$\begin{aligned}
B(-x_{12}, x_{22}^{-1})&(x_{12}-x_{22}+x_{33})\\
&= x_{12}-x_{22}+x_{33}+\{x_{12}, x_{22}^{-1}, x_{12}-x_{22}+x_{33}\}\\
&\quad +Q(x_{12})\,Q(x_{22}^{-1})(x_{12}-x_{22}+x_{33})\\
&= x_{12}-x_{22}+x_{33}+2Q(x_{12})\,x_{22}^{-1}-x_{12}-Q(x_{12})\,x_{22}^{-1}\\
&= x_{11}-x_{22}+x_{33}.
\end{aligned}$$

Remark. This calculation reflects the well-known fact that the symmetric matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are congruent over the integers. Similarly, Proposition 1 corresponds to the matrix equation

$$^tA \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

THEOREM 1. *A nonzero element x of a simple nondegenerate Jordan pair with principal dcc is diagonalizable if and only if $Q_y x \neq 0$ for some simple element y.*

Proof. If $x = \sum_{i=1}^r x_{ii}$ is diagonal with respect to $e_1, ..., e_r$ and (say) $x_{11} \neq 0$ then e_1^- is a simple element and $x_{11} = Q(e_1^+) Q(e_1^-) x$ shows $Q(e_1^-) x \neq 0$. Conversely, suppose $Q_y x \neq 0$ for a simple y, and extend $y = e_-$ to a division idempotent e. Then the Peirce-2-component $x_2 = Q(e_+) Q(e_-) x$ of x relative to e is not zero since $Q(e_-) x_2 = Q(e_-) x = Q_y x$, and hence it is invertible in $V_2(e)$. By Lemma 1 of [6], we may replace x by an elementary transform and then assume $x = x_2 + x_0$ relative to e. Applying Lemma 1 to $x_0 \in V_0^+(e)$ we see that x has the weak diagonal form indicated there, with $t > 0$. If $s = 0$ we are done. Otherwise apply Lemma 2 to $x_{12} + x_{2s+1,2s+1}$. This is possible since V is simple and therefore connected [7, Theorem 2]. The result is an elementary transform x' having again weak diagonal form but with $s' = s - 1$ and $t' = t + 2$. Now the theorem follows by induction.

This motivates the following definition. Call an element x of a Jordan pair V *defective* if $Q_y x = 0$ for all simple elements y. The *defect* of V is $\operatorname{Def} V = (\operatorname{Def} V^+, \operatorname{Def} V^-)$ where $\operatorname{Def} V^\sigma$ denotes the set of defective elements in V^σ. For the relation with the defect of a quadratic form, see the examples after Theorem 2.

COROLLARY 1. *Let V be nondegenerate with principal dcc and $0 \neq x \in V^+$. Then there exists a unique decomposition $V = V' \oplus V''$ (direct* *sum of ideals) such that, if $x = x' + x''$ is decomposed accordingly, x' is diagonalizable and x'' is defective.*

This is an immediate consequence of the decomposition of V into simple ideals [7, Theorem 2] which is clearly compatible with the defect.

COROLLARY 2. *For a nondegenerate Jordan pair with principal dcc the following conditions are equivalent.*

(i) Def $V = 0$;

(ii) *every element is diagonalizable*;

(iii) *every idempotent is a sum of orthogonal division idempotents.*

Proof. (i)$\Leftrightarrow$(ii) is immediate from Corollary 1. For (ii)$\Rightarrow$(iii), let c be an idempotent and let $c_+ = x_{11} + \cdots + x_{rr}$ be diagonal with respect to $e_1, ..., e_r$. Omitting those e_i with $x_{ii} = 0$ we may assume all $x_{ii} \neq 0$. Then x_{ii} is invertible in V_{ii}, and $d_i = (x_{ii}, x_{ii}^{-1})$ are orthogonal division idempotents such that c and $d = d_1 + \cdots + d_r$ have the same V^+-component. By [6], Corollary of Proposition 2, c and d are conjugate by an elementary automorphism, proving (iii). Finally, every x can be extended to an idempotent by regularity, which shows (iii)$\Rightarrow$(ii).

2. STRUCTURE OF THE DEFECT

Recall that a *structural transformation* (f, g): $V \rightleftarrows W$ between Jordan pairs is a pair of k-linear maps f: $V^+ \to W^+$ and g: $W^- \to V^-$ such that $Q(f(x)) = fQ_x g$ and $Q(g(y)) = gQ_y f$, for all $x \in V^+$ and $y \in W^-$.

PROPOSITION 2. *Let V and W be arbitrary Jordan pairs.*

(a) *If (f, g): $V \rightleftarrows W$ is structural then f(Def V^+) $\subset$ Def W^+ and g(Def W^-) $\subset$ Def V^-.*

(b) Def *V is an outer ideal of V, stable under all structural transformations.*

(c) *If c is an idempotent of V then* Def $V_j(c) = V_j(c) \cap$ Def V, *for $j = 0, 2$, and* Def *V is stable under all Peirce projections.*

Proof. (a) It suffices to prove the first inclusion since (g, f): $W^{op} \rightleftarrows V^{op}$ is again structural. Let $x \in$ Def V^+. If $u = f(x)$ were not defective then $v' = Q_v u \neq 0$ for some simple $v \in W^-$. Now v generates a simple inner ideal $M = Q_v W^+$ of W^-, and by [7, Lemma 2], $g(M)$ is either a simple or a trivial inner ideal of V^-. In particular, $g(v)$ is either a simple or a trivial element of V^-, proving $Q(g(v))\, x = 0$. It follows that

$Q_u Q_v u = Q(f(x)) Q_v f(x) = (fQ_x gQ_v f)(x) = fQ_x Q(g(v)) x = 0$, hence $Q_{v'} u = Q_v Q_u Q_v u = 0$ as well. But $0 \neq v' \in M$ implies $M = Q_{v'} W^+$ by simplicity of M. Hence $v = Q_{v'} w$ for some $w \in W^+$, and $v' = Q_v u = Q(Q_{v'} w) u = Q_{v'} Q_w Q_{v'} u = 0$, a contradiction.

(b) Since (Q_z, Q_z) and $(B(x, y), B(y, x))$ are structural transformations, this follows from (a).

(c) Let $i: V_j^+(c) \to V^+$ be the inclusion and $p: V^- \to V_j^-(c)$ the Peirce projection. Then $(i, p): V_j(c) \rightleftarrows V$ is structural by the Peirce relations. Hence $\operatorname{Def} V_j^+(c) \subset \operatorname{Def} V^+$ and $p(\operatorname{Def} V^-) \subset \operatorname{Def} V_j^-(c)$ by (a), which implies $\operatorname{Def} V_j^+(c) \subset V_j^+(c) \cap \operatorname{Def} V^+$ and $V_j^-(c) \cap \operatorname{Def} V^- \subset \operatorname{Def} V_j^-(c)$. By interchanging the roles of $+$ and $-$, the first part of the assertion follows. For the second, let $x = x_2 + x_1 + x_0$ be the Peirce decomposition of $x \in \operatorname{Def} V^\sigma$. Then $x_2, x_0 \in \operatorname{Def} V^\sigma$ since the diagonal Peirce projections are structural, and so $x_1 \in \operatorname{Def} V^\sigma$ as well.

COROLLARY. *Let $W = \operatorname{Def} V$ and $c \in W$ an idempotent. Then $W_1(c) = V_1(c)$.*

Proof. This holds in fact for any outer ideal. Clearly $W_1(c) \subset V_1(c)$. Conversely, if $x \in V_1^\sigma(c)$ then $x = \{c_\sigma c_{-\sigma} x\} \in Q(c_\sigma, x) W^{-\sigma} \subset W^\sigma$ by outerness of W.

LEMMA 3. Def $V = 0$ *when V is nondegenerate of capacity* 1.

Proof. By [6, Th. 3], every nonzero element is simple, hence part of a division idempotent. In particular, every element is diagonalizable and thus $\operatorname{Def} V = 0$.

LEMMA 4. *Let V be nondegenerate of capacity 2, decompose*

$$V = V_{11} \oplus V_{12} \oplus V_{22} \oplus V_{10} \oplus V_{20}$$

relative to a frame $F = \{e_1, e_2\}$, and set $e = e_1 + e_2$.

(a) Def $V = V_{12}^\natural$ *is the radical of the bilinear form $\Phi(x, y) = \{x, y, e_+\}$ on $V_{12}^+ \times V_{12}^-$ with values in $V_{11}^+ \oplus V_{22}^+$.*

(b) *Every nonzero element of* Def V *is invertible in V; in particular, if* Def $V \neq 0$ *then $V_{10} = V_{20} = 0$ and* Def V *is a division pair.*

(c) *The elementary group $E(V)$ acts like the identity on* Def V.

Proof. Let $0 \neq x = x_{11} + x_{12} + x_{22} + x_{10} + x_{20} \in V^+$ be defective. Then $x_{11} + x_{10}$ is defective in the capacity 1 Jordan pair $V_{11} \oplus V_{10} = V_0(e_2)$ by Proposition 2 and thus vanishes by Lemma 3. Similarly, $x_{22} + x_{20} = 0$ so $x = x_{12} \in V_{12}^+$. For any $y \in V_{12}^-$ the elementary transform $x' = B(e_i^+, -y) x$ is again defective, and by the Peirce relations, the V_{ii}^+-component of x' is $\{x, y, e_i^+\}$. Hence $\Phi(x, y)=0$ for all $y \in V_{12}^-$. By [6, Lemma 2], $x \neq 0$ implies x invertible in $V_2(e)$.

We show next that $x = x_{12}$ is in fact invertible in V or, what amounts to the same, that $V_1(e) = V_{10} \oplus V_{20} = 0$. Let $y_{12} \in V_{12}^-$ be the relative inverse of x_{12} in $V_2(e)$. Then $y_{12} = Q(y_{12})\, x_{12} \in \operatorname{Def} V^-$ by outerness of the defect, hence $c = (x_{12}, y_{12}) \in \operatorname{Def} V$ is an idempotent associated to e. By the Corollary of Proposition 2 we have $V_1(c) = V_1(e) \subset \operatorname{Def} V$. But we have seen above that $\operatorname{Def} V \subset V_{12}$ whence $V_1(e) = 0$. This proves (b).

Before showing $V_{12}^\natural \subset \operatorname{Def} V$ we prove (c). We may assume $V_{10} = V_{20} = 0$. Let $x \in V_{12}^{+\natural}$ and $(u, v) \in V_{12} = V_1(e_i)$. Then $B(e_i^+, v)\, x = x - \{e_i^+, v, x\} = x$ by the Peirce relations and the assumption on x. Furthermore, $B(u, e_i^-)\, x = x - \{u, e_i^-, x\}$, and from the identity JP9 [5, p. 14] it follows easily that $\{u, e_i^-, x\} = \{x, Q(e_-)\, u, e_{3-i}^+\} = 0$. Thus the generators of $E(F, V) = E(V)$ [6, Corollary 1 of Th. 2]) act trivially on $V_{12}^{+\natural}$, and (c) holds.

Now let d be a division idempotent. Then $\varphi(d) \approx e_i$ for some $\varphi \in E(V)$ by [6, Th. 2], and $\varphi_+(x) = x$, showing $\varphi_-(Q(d_-)\, x) \in Q(V_{ii}^-)\, V_{12}^+ = 0$ by the Peirce relations. This proves $V_{12}^{+\natural} \subset \operatorname{Def} V^+$. Finally, all arguments hold for V^- in place of V^+ as well since $\{y, x, e_-\} = Q(e_-)\, \Phi(x, y)$ so the radical of Φ is the same as that of the bilinear form $\{y, x, e_-\}$. This completes the proof.

LEMMA 5. *Let V be a nondegenerate Jordan pair with principal dcc. Then $x \in \operatorname{Def} V^+$ if and only if there exist orthogonal division idempotents $e_1, \ldots, e_{2s}$ such that*

$$x = \sum_{i=1}^{s} x_{2i-1,2i},$$

where $0 \neq x_{2i-1,2i} \in V_{2i-1,2i}^{+\natural}$ is in the defect of the capacity 2 Jordan pair $V_{2i-1,2i-1} \oplus V_{2i-1,2i} \oplus V_{2i,2i} = V_2(e_{2i-1} + e_{2i})$.

Proof. Write x in weak diagonal form as in Lemma 1. Then by Proposition 2 and Lemmas 3 and 4, x is defective if and only if $t = 0$ and $x_{2i-1,2i} \in \operatorname{Def} V_2^+(e_{2i-1} + e_{2i}) = V_{2i-1,2i}^{+\natural}$.

THEOREM 2. *If V is a nondegenerate Jordan pair with principal dcc so is* $\operatorname{Def} V$, *and* $\operatorname{Def}(\operatorname{Def} V) = 0$. *If, in addition, V is simple and* $\operatorname{Def} V \neq 0$ *then* $\operatorname{Def} V$ *is simple and $2V = 0$. Furthermore, in this case V has finite capacity if and only if* $\operatorname{Def} V$ *does, and then* $\kappa(\operatorname{Def} V) = [\frac{1}{2}\kappa(V)]$.

Proof. Being an outer ideal, $W = \operatorname{Def} V$ is regular along with V, hence in particular nondegenerate. If we write an element $x \in W^+$ as in Lemma 5 then if follows from Lemma 4 and the Peirce relations that $x_{2i-1,2i}$ is a simple element of W. Thus $W = \operatorname{Soc} W$, proving W has principal dcc by [7, Corollary 1 of Th. 1]. If $y_{2i-1,2i} \in V^{-}_{2i-1,2i}$ is the relative inverse of $x_{2i-1,2i}$ in $V_2(e_{2i-1}+e_{2i})$ then the $f_i = (x_{2i-1,2i}, y_{2i-1,2i})$ are orthogonal division idempotents of W, associated to $e_{2i-1}+e_{2i}$. Thus every element of W is diagonalizable in W, proving $\operatorname{Def} W = 0$ by Corollary 2 of Th. 1.

Now let V be simple. To show W is simple as well it suffices to have $W_1(c) \cap W_1(d) \neq 0$ for any two orthogonal division idempotents c and d of W [7, Lemma 6, Th. 2]. But this follows from the Corollary of Proposition 2 and connectedness of V. Since $2V$ is an ideal of V, Proposition 1 and Th. 1 show $2V = 0$.

Finally, let $\kappa(V) = n$ be finite and $\{e_1, ..., e_n\}$ a frame of V. For $i = 1, ..., m = [n/2]$ choose nonzero $x_{2i-1,2i} \in V^{+\natural}_{2i-1,2i}$ and define f_i as above. Then $F = \{f_1, ..., f_m\}$ is a set of orthogonal division idempotents of W, and $W_0(F) = 0$: Indeed, $W_0(F) = W_0(f_1 + \cdots + f_m) = W \cap V_0(f_1 + \cdots + f_m) = W \cap V_0(e_1 + \cdots + e_{2m})$ (since f_i is associated to $e_{2i-1}+e_{2i}) = \operatorname{Def} V_0(e_1 + \cdots + e_{2m})$ (by Proposition 2(c)) $= 0$ since $\kappa(V_0(e_1 + \cdots + e_{2m})) = n - 2m \leqslant 1$. Thus F is a frame of W and $\kappa(W) = m$. On the other hand, if $\kappa(V)$ is infinite then there exists an infinite set $\{e_1, e_2, ...\}$ of orthogonal division idempotents of V giving rise, by the above construction, to an infinite set of orthogonal division idempotents of W, and thus $\kappa(W)$ is infinite too. This completes the proof.

EXAMPLES. We determine the defect explicitly for the Jordan pairs described in the structure theorems [5, 12.12; 7, Theorem 3]. Thus let V be a simple Jordan pair with principal dcc and $\operatorname{Def} V \neq 0$.

(a) Suppose V has finite capacity, and let $V = \sum_{0 \leqslant i \leqslant j \leqslant n} V_{ij}$ be the Peirce decomposition of V with respect to some frame $\{e_1, ..., e_n\}$. By Lemma 3, $V_{i0} = 0$ for $i = 1, ..., n$. Thus V contains invertible elements. There are two cases.

(i) V is the Jordan pair of a Jordan algebra $H_n(R, j, R_0)$ of hermitian matrices where the possible coordinate algebras $C = (R, j, R_0)$ are listed in [3, 6.4.1, case III]. Here $V_{12} \cong R$ and $\{x, y, e_i^+\} \in V^+_{ii}$ corresponds to $x\bar{y} + y\bar{x} \in R_0$. It is an easy exercise to show that $V^{\natural}_{12} \neq 0$ if and only if $R = K$ is a commutative field of characteristic 2 and j is the identity. Hence in this case $V^{\natural}_{12} = V_{12}$ and $\operatorname{Def} V$ is the Jordan pair of alternating (= symmetric with zero diagonal since the characteristic is 2) $n \times n$ matrices over K.

(ii) V is an outer ideal in a Jordan pair of a quadratic form. Since V is not a division pair, it follows from [9, Sect. 3] that it has the following structure: $V^{\pm} = K_0 \cdot u \oplus K_0 \cdot v \oplus M$ where K is a field of characteristic 2, M

is a K-vectorspace with a nondegenerate quadratic form q representing 1, and $K_0 \subset K$ is an additive subgroup containing 1 and satisfying $q(M)\,K_0 \subset K_0$. Extend q to a quadratic form on $X = K\cdot u \oplus K\cdot v \oplus M$ by $q(\lambda u \oplus \mu v \oplus w) = \lambda\mu + q(w)$ and let $b(x, y) = q(x+y) - q(x) - q(y)$ be the associated bilinear form. Then the Jordan product on V is given by $Q_x y = b(x, y)\,x - q(x)\,y$. Also, $e_1 = (u, v)$, $e_2 = (v, u)$ is a frame of V with $V_{ii} \cong (K_0, K_0)$ and $V_{12}^{\pm} = M$, and $\{x, y, e_i^+\} = b(x, y)\cdot e_i^+$ for $(x, y) \in V_{12}^+ \times V_{12}^- = M \times M$. Hence Def $V^{\pm} = M^{\perp} = \{x \in M : b(x, M) = 0\}$ is the defect of the quadratic form q on M (cf. [2, 8.1]). Note, however, that the Jordan pair of an anisotropic quadratic form is a division pair and thus has defect zero although the quadratic form may be defective.

(b) If V has infinite capacity then it is the direct limit of the finite capacity subpairs $V_2(e)$ where e runs through the idempotents of V, and $V_2(e)$ is the Jordan pair of $H_n(C)$ for an associative coordinate algebra C. By Proposition 2(c) and case (i) above we have $C = (K, \mathrm{Id}, K_0)$ where K is a field of characteristic 2, and Def $V = \varinjlim$ Def $V_2(e)$ is a direct limit of alternating matrices over K.

3. The Rank Function

Let V be a nondegenerate Jordan pair. For an element $x \in V^{\sigma}$ let $\mathrm{rk}(x)$, the *rank* of x, denote the supremum of the lengths of all finite chains $[x_0] \subset [x_1] \subset \cdots \subset [x_n]$ of principal inner ideals $[x_i] = Q(x_i)\,V^{-\sigma}$ where x_i belongs to the inner ideal $(x) = k\cdot x + [x]$ generated by x, and the length of such a chain is the number of strict inclusions. Clearly, $\mathrm{rk}(x)$ is a nonnegative integer or ∞ and depends only on (x). Also note that $x_i = \lambda x + Q_x y \in (x)$ implies $[x_i] \subset [x]$ by the identity $\{x, y, Q_x z\} = Q_x\{y, x, z\}$. Hence a chain of length $n = \mathrm{rk}(x)$ must have $x_0 = 0$ and $[x_n] = [x]$.

In the standard examples of Jordan pairs of rectangular or hermitian matrices over division rings, the rank defined here is just the matrix rank, whereas for alternating matrices over a field it is one-half the matrix rank. A nonzero element x of a Jordan pair of a nondegenerate isotropic quadratic form q over a field has rank one or two depending on whether $q(x) = 0$ or $q(x) \neq 0$. It should be noted also that for Jordan pairs over a field, the notion of rank one element as defined in [8] is more restrictive than the present one.

Two elements u, $x \in V^{\sigma}$ are called *orthogonal* ($u \perp x$) if $u = c_{\sigma}$, $x = d_{\sigma}$ are part of orthogonal idempotents $c = (c_+, c_-)$ and $d = (d_+, d_-)$. For example, if $V = V' \oplus V''$ is the direct sum of two regular ideals and $x = x' + x''$ decomposes correspondingly then $x' \perp x''$.

PROPOSITION 3. *The rank function of a nondegenerate Jordan pair has the following properties.*

(1) $\mathrm{rk}(x)=0 \Leftrightarrow x=0$.

(2) $\mathrm{rk}(x)<\infty \Leftrightarrow x$ *is in the socle. In this case, x is regular, and if $x=c_\sigma$ is part of an idempotent c then* $\mathrm{rk}(x)=\kappa(V_2(c))$.

(3) $\mathrm{rk}(x)=1 \Leftrightarrow x$ *is a simple element.*

(4) *If* $(x)\subset(u)$ *then* $\mathrm{rk}(x)\leqslant \mathrm{rk}(u)$.

(5) $\mathrm{rk}(Q_x y)\leqslant \mathrm{rk}(x)$.

(6) *If* $(x)\subset(u)$ *and* $\mathrm{rk}(x)=\mathrm{rk}(u)<\infty$ *then* $(x)=(u)$.

(7) *If* $u\perp x$ *then* $\mathrm{rk}(u+x)=\mathrm{rk}(u)+\mathrm{rk}(x)$.

Proof. (1) follows from nondegeneracy, and (2) from [7, Th. 1]. A simple element x generates a simple inner ideal $(x)=[x]$ and thus has rank one. Conversely, a rank one element x is regular by (2), and therefore $(x)=[x]$ is a simple inner ideal by the rank one condition, proving x simple. (4) is immediate from the definition and (5) from (4) and $(Q_x y)\subset(x)$. By (2), elements x and u of finite rank are regular, so $[x]=(x)$ and $[u]=(u)$. If $[x]$ is properly contained in $[u]$ then $\mathrm{rk}(u)\geqslant \mathrm{rk}(x)+1$, proving (6). Finally, let $u\perp x$. If $\mathrm{rk}(u+x)=\infty$ then either $\mathrm{rk}(u)$ or $\mathrm{rk}(x)$ is infinite by (2) and there is nothing to prove. Thus assume $\mathrm{rk}(u+x)$ finite, and $u=c_\sigma, x=d_\sigma$ for orthogonal idempotents c and d. Then $u=Q(u+x)\,c_{-\sigma}$ has finite rank by (5), and similarly $\mathrm{rk}(x)<\infty$. Let $W=V_2(c+d)$. Then $W_2(c)=V_2(c)$ and $W_0(c)=V_2(d)$. Hence by [6, Lemma 5] $\mathrm{rk}(u+x)=\kappa(W)=\kappa(W_2(c))+\kappa(W_0(c))=\kappa(V_2(c))+\kappa(V_2(d))=\mathrm{rk}(u)+\mathrm{rk}(x)$.

COROLLARY 1. *If $V=\mathrm{Soc}\,V$ is nondegenerate and contains invertible elements then $\kappa(V)=n$ is finite, and an element x is invertible if and only if* $\mathrm{rk}(x)=n$.

Proof. In a Jordan pair containing invertible elements, $x\in V^\sigma$ is invertible if and only if $[x]=V^\sigma$. Now the corollary follows from (6) and [6, Th. 3].

COROLLARY 2. *Let $x\in\mathrm{Soc}\,V^+$ be written in weak diagonal form as in Lemma* 1. *Then* $\mathrm{rk}(x)=2s+t$.

This is clear from (7) and Corollary 1.

LEMMA 6. *Let V be nondegenerate.*

(a) *If $U\subset V$ is a full subpair then the rank function of U is the restric-*

tion of the rank function of V to U. In particular, this holds for the socle Soc V.

(b) *Let* $W = \mathrm{Def}(\mathrm{Soc}\, V)$. *Then* $\mathrm{rk}_W(x) = \frac{1}{2}\mathrm{rk}(x)$ *for* x *in* W.

Proof. (a) A full subpair has $Q_x U^{-\sigma} = Q_x V^{-\sigma}$ for all $x \in U^\sigma$ (cf. [7, Sect. 1]), hence (a) follows immediately from the definition. The socle is a regular ideal and thus a full subpair.

(b) By (a), we may assume $V = \mathrm{Soc}\, V$ has principal dcc. Representing x as in Lemma 5, we have $\mathrm{rk}(x) = 2s$, and $x_{2i-1,2i}$ is a simple element of W hence $\mathrm{rk}_W(x) = s$.

THEOREM 3. *Let V and W be nondegenerate Jordan pairs.*

(a) $\mathrm{rk}(u+x) \leqslant \mathrm{rk}(u) + \mathrm{rk}(x)$, *for all* $u, x \in V^\sigma$. *If u and x have finite rank then equality holds if and only if* $u \perp x$.

(b) *Let* $(f, g)\colon V \rightleftarrows W$ *be structural. Then* $\mathrm{rk}(f(x)) \leqslant \mathrm{rk}(x)$ *and* $\mathrm{rk}(g(y)) \leqslant \mathrm{rk}(y)$, *for* $x \in V^+$, $y \in W^-$.

Proof. (a) We may assume $\sigma = +$. If either u or x has infinite rank there is nothing to prove. If u and x have finite rank we may replace V by its socle so V is nondegenerate with principal dcc, and by decomposing V into simple summands we may even assume V simple (cf. the remark preceding Proposition 3). First suppose $\mathrm{rk}(x) = 1$; extend $u = c_+$ to an idempotent c, and let $x = x_2 + x_1 + x_0$ be the Peirce decomposition of x relative to c. We distinguish the following cases.

Case 1. $x_0 \neq 0$. Then by [7, Lemma 4], $u + x = c_+ + x$ is conjugate to $c_+ + d_+$ by an automorphism fixing c_+ where d is division idempotent orthogonal to c. Since the rank function is clearly invariant under automorphisms, $\mathrm{rk}(u+x) = \mathrm{rk}(c_+ + d_+) = \mathrm{rk}(c_+) + \mathrm{rk}(d_+) = \mathrm{rk}(u) + 1 = \mathrm{rk}(u) + \mathrm{rk}(x)$, and we also have $u \perp x$.

Case 2. $x_1 \neq 0 = x_0$. Then $u + x$ is conjugate to u by [7, Lemma 5], and therefore $\mathrm{rk}(u+x) = \mathrm{rk}(u)$.

Case 3. $x_0 = x_1 = 0$. Then $u + x = c_+ + x_2 \in V_2^+(c) = (u)$ and hence $\mathrm{rk}(u+x) \leqslant \mathrm{rk}(u)$ by (4) of Proposition 3.

Next, if $\mathrm{rk}(x) = n$ and x is diagonalizable then x is an orthogonal sum of n rank one elements, and the triangle inequality follows by an obvious induction. To prove the second statement of (a) in this case, write $x = x' + x''$ where $x' \perp x''$ and $\mathrm{rk}(x') = 1$. Then $\mathrm{rk}(u) + \mathrm{rk}(x'') + 1 = \mathrm{rk}(u) + \mathrm{rk}(x) = \mathrm{rk}(u + x)$ (by hypothesis) $= \mathrm{rk}((u + x'') + x') \leqslant \mathrm{rk}(u + x'') + 1 \leqslant \mathrm{rk}(u) + \mathrm{rk}(x'') + 1$ (by induction) implies $\mathrm{rk}(u + x'') = \mathrm{rk}(u) + \mathrm{rk}(x'')$ and $\mathrm{rk}(u' + x') = \mathrm{rk}(u') + \mathrm{rk}(x')$ where $u' = u + x$. By what we proved above, $u' \perp x'$ hence $u' \in V_0^+(d')$ where d' is a division idempotent with $d'_+ = x'$. By induction, $u \perp x''$ whence $u, x \in [u'] \subset V_0^+(d')$ since $V_0(d')$ is a full subpair. Therefore u and x'' can be extended to orthogonal idempotents in $V_0(d')$, and it follows easily that u and $x = x' + x''$ are orthogonal.

If x is not diagonalizable but u is then we interchange the roles of x and u. This leaves the case where neither x nor u is diagonalizable. But then they are defective by Th. 1 and hence are diagonalizable in Def V by Th. 2. Now the assertion follows from Lemma 6(b).

(b) Inasmuch as (g, f) is again structural it suffices to prove the first inequality. If x has infinite rank there is nothing to prove. By [7, Proposition 2], we may therefore assume $V = \text{Soc } V$ and $W = \text{Soc } W$. If $x = x' + x''$ is an orthogonal sum and (b) holds for the constituents then it holds for x as well, by (a) and additivity of f. This reduces us to the case where x is either diagonalizable or defective (Corollary 1 of Th. 1). If x has rank one then it is simple and by [7, Lemma 2], $f(x)$ is either simple or zero, proving $\text{rk}(f(x)) \leqslant \text{rk}(x)$ in this case. Thus if x is diagonalizable we are done, and if it is defective then it is diagonalizable in Def V and the assertion follows from Proposition 2 and Lemma 6.

Applying (b) to the inner structural transformations, we get

COROLLARY 1. (a) $\text{rk}(Q_x y) \leqslant \min(\text{rk}(x), \text{rk}(y))$, *and* $\text{rk}(B(x, y)\, z) \leqslant \text{rk}(z)$. *In particular*, $\text{rk}(c_+) = \text{rk}(c_-)$ *for any idempotent c.*

(b) *If* $x = x_2 + x_1 + x_0$ *is the Peirce decomposition of an element x relative to some idempotent then* $\max(\text{rk}(x_2), \text{rk}(x_0)) \leqslant \text{rk}(x)$.

COROLLARY 2. *Let* $\text{rk}(x) = r < \infty$. *Then x is diagonalizable if and only if it is the sum of r rank one elements.*

This follows from (a) of Theorem 3, and naturally raises the question: *How many rank one elements are needed to represent a defective element of rank r?* The answer is

COROLLARY 3. *Let V be simple and x a defective element of rank* $r = 2s > 0$. *Then x is the sum of* $r + 1$ *elements of rank one.*

Proof. Write $x = x_{12} + x_{34} + \cdots + x_{2s-1,2s}$ as in Lemma 5. Then x_{12} is the sum of three simple elements: Indeed, let $u = B(-x_{12}, e_1^-)\, e_1^+ = e_1^+ + x_{12} + Q(x_{12})\, e_1^- \in V_{11}^+ \oplus V_{12}^+ \oplus V_{22}^+$. Then $\text{rk}(u) = \text{rk}(Q(x_{12})\, e_1^-) = 1$ by Corollary 1 (observe $Q(x_{12})\, e_1^- \neq 0$ since x_{12} is invertible in $V_{11} \oplus V_{12} \oplus V_{22}$). Hence $x_{12} = u - e_1^+ - Q(x_{12})\, e_1^-$ is the sum of three elements of rank one. Now let $x' = u + x_{34} + \cdots + x_{2s-1,2s}$. Then $Q(e_1^-)\, x' = e_1^- \neq 0$. By Theorem 1, x' is diagonalizable, hence a sum of $\mathrm{rk}(x') = 1 + 2s - 2 = 2s - 1$ simple elements. It follows that $x = -e_1^+ - Q(x_{12})\, e_1^- + x'$ is a sum of $2s+1$ simple elements.

In view of the examples after Theorem 2, this says in particular:

COROLLARY 4. *An alternating matrix of rank r over a field of characteristic 2 is the sum of $r+1$ but not of r symmetric matrices of rank one.*

We now prove a rank inequality which may be considered the Jordan analogue of the *Frobenius inequality*

$$\mathrm{rk}(xy) + \mathrm{rk}(yz) \leqslant rk(xyz) + \mathrm{rk}(y)$$

for matrices. It was M. Koecher who noted that this is equivalent to the inequality

$$\mathrm{rk}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \mathrm{rk}\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \leqslant \mathrm{rk}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{rk}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for block matrices (cf. [4, 2.6.4]). Since

$$\mathrm{rk}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \leqslant \mathrm{rk}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \mathrm{rk}\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$$

is immediate from looking at the row spaces, we have

$$\mathrm{rk}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \leqslant \mathrm{rk}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{rk}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and it is in this weaker form that the Frobenius inequality makes sense in the Jordan setting.

Recall from [10, (1.3)] that a *Peirce grading* of a Jordan pair V is a decomposition $V^\sigma = V_0^\sigma \oplus V_1^\sigma \oplus V_2^\sigma$ ($\sigma = \pm$) satisfying the multiplication rules for Peirce spaces but not necessarily arising from a Peirce decomposition with respect to an idempotent. A typical example is a block decomposition of matrices, with V_0^σ, V_1^σ, V_2^σ consisting of all matrices of the forms

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix},$$

respectively. Then

$$x_0 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in V_0^+, \qquad x_1 = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in V_1^+, \qquad x = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and the inequality becomes

$$\mathrm{rk}(x_0+x_1) \leqslant \mathrm{rk}(x_0)+\mathrm{rk}(x),$$

which is true in general (cf. Theorem 4 below). We first prove a lemma.

LEMMA 7. *Let V be nondegenerate with principal dcc, and let $x=x_1+x_2$ be the decomposition of an element $x\in V^+$ with respect to a Peirce grading. Then*

$$\mathrm{rk}(x_1+\lambda x_2) \leqslant \mathrm{rk}(x_1+x_2)$$

for all $\lambda\in k$.

Proof. By regularity, $x_1=Q(x_1)\,y$ for some $y\in V^-$, and decomposing $y=y_2+y_1+y_0$ we see $x_1=Q(x_1)\,y_1$. Hence $x_1=e_+$ is part of an idempotent $e=(e_+,e_-)=(x_1,Q(y_1)\,x_1)\in V_1$. From the multiplication rules for a Peirce grading it is immediate that the V_i are stable under the Peirce projections relative to e, and we have a simultaneous decomposition $V=\bigoplus V_{(ij)}$ where $V_{(ij)}=V_i\cap V_j(e)$. Decompose $x_2=x_{(22)}+x_{(21)}+x_{(20)}$ accordingly. Then an e-elementary transform of $x_1+\lambda x_2$ is

$$\begin{aligned} B(\lambda x_{(21)}, e_-)(\lambda x_2+x_1) &= \lambda x_2 + B(\lambda x_{(21)}, e_-)\,e_+ \\ &= \lambda x_2 + e_+ - \lambda x_{(21)} + Q(\lambda x_{(21)})\,e_- \\ &= (e_+ + \lambda x_{(22)}) + \lambda x_{(20)}. \end{aligned}$$

Here $e_+ + \lambda x_{(22)}\in V_2^+(e)$ is orthogonal to $\lambda x_{(20)}\in V_0^+(e)$. Hence by (7) of Proposition 3,

$$\mathrm{rk}(x_1+\lambda x_2)=\mathrm{rk}(e_+ + \lambda x_{(22)})+\mathrm{rk}(\lambda x_{(20)}).$$

Now $x_{(22)}\in Q(e_+)\,V^-$, and by the multiplication rules for the $V_{(ij)}$, we have $x_{(22)}=Q(e_+)\,y_{(02)}$ for some $y_{(02)}\in V^-_{(02)}$. A computation shows

$$Q(e_+ + \alpha x_{(22)})(e_- + \beta y_{(02)}) = e_+ + (2\alpha+\beta)\,x_{(22)}$$

for all α, $\beta\in k$. Consequently, $[e_+ + (2\alpha+\beta)\,x_{(22)}]\subset[e_+ + \alpha x_{(22)}]$, therefore the principal inner ideal $[e_+ + \lambda x_{(22)}]$ is independent of λ, and so is $\mathrm{rk}(e_+ + \lambda x_{(22)})$. On the other hand, multiplication by λ is structural whence $\mathrm{rk}(\lambda x_{(20)})\leqslant \mathrm{rk}(x_{(20)})$ by Th. 3. This shows $\mathrm{rk}(x_1+\lambda x_2)\leqslant \mathrm{rk}(e_+ + x_{(22)})+\mathrm{rk}(x_{(20)})=\mathrm{rk}(x_1+x_2)$ by the computation above, specialized to the case $\lambda=1$.

THEOREM 4. *Let $V=V_0\oplus V_1\oplus V_2$ be a Peirce grading of a non-*

degenerate Jordan pair V with principal dcc, and let $x = x_0 + x_1 + x_2$ be the corresponding decomposition of an element $x \in V^+$. Then

$$\operatorname{rk}(x_0 + x_1) \leqslant \operatorname{rk}(x_0) + \operatorname{rk}(x).$$

Proof. Extend $x_0 = c_+$ to an idempotent c which, by a similar argument as above, may be chosen in V_0. Then the Peirce decomposition relative to c is compatible with the given Peirce grading, and since $c \in V_0$ we get $V_2(c) \subset V_0$, $V_1^\sigma(c) = \{c_\sigma c_{-\sigma} V_1^\sigma(c)\} \subset \{V_0^\sigma V_0^{-\sigma} V^\sigma\} \subset V_0^\sigma \oplus V_1^\sigma$, so $V_{(12)} = V_{(22)} = V_{(21)} = 0$ and

$$V_2 = V_{(20)}, \qquad V_1 = V_{(11)} \oplus V_{(10)}, \qquad V_0 = V_{(02)} \oplus V_{(01)} \oplus V_{(00)},$$

where $V_{(ij)} = V_i \cap V_j(c)$. Thus $x_2 = x_{(20)}$, $x_1 = x_{(11)} + x_{(10)}$, and $x_0 = x_{(02)}$ by definition of c. Then a c-elementary transform of $x_0 + x_1$ is

$$\begin{aligned}
&B(x_{(11)}, c_-)(x_0 + x_1)\\
&\quad = B(x_{(11)}, c_-)\, c_+ + B(x_{(11)}, c_-)(x_{(11)} + x_{(10)})\\
&\quad = c_+ - x_{(11)} + Q(x_{(11)})\, c_-\\
&\qquad + x_{(11)} + x_{(10)} - \{x_{(11)}, c_-, x_{(11)} + x_{(10)}\} + Q(x_{(11)})\, Q(c_-)\, x_1\\
&\quad = c_+ + (x_{(10)} - Q(x_{(11)})\, c_-)\\
&\quad = x_0 + (x_{(10)} - u_{(20)}),
\end{aligned}$$

by the multiplication rules for the $V_{(ij)}$. Here $u_{(20)} = Q(x_{(11)})\, c_- \in V^+_{(20)}$ and therefore $x_{(10)} - u_{(20)} \in V_0^+(c)$ is orthogonal to $x_0 = c_+$ which implies

$$\operatorname{rk}(x_0 + x_1) = \operatorname{rk}(x_0) + \operatorname{rk}(x_{(10)} - u_{(20)}). \tag{$*$}$$

Since $c_- \in V_0^-$ and $x_2 \in V_2^+$, we have $B(x_{(11)}, c_-)\, x_2 = x_2$ whence $B(x_{(11)}, c_-)\, x = x_0 + (x_{(10)} + (x_{(20)} - u_{(20)}))$ and, since $x_0 \in V_2^+(c)$ is orthogonal to $x_{(10)} + (x_{(20)} - u_{(20)}) \in V_0^+(c)$,

$$\operatorname{rk}(x) = \operatorname{rk}(x_0) + \operatorname{rk}(x_{(10)} + (x_{(20)} - u_{(20)})). \tag{$**$}$$

By the triangle inequality, $\operatorname{rk}(x_{(10)} - u_{(20)}) \leqslant \operatorname{rk}(x_{(10)}) + \operatorname{rk}(u_{(20)})$, and by Corollary 1 of Th. 3, $\operatorname{rk}(u_{(20)}) = \operatorname{rk}(Q(x_{(11)})\, c_-) \leqslant \operatorname{rk}(c_-) = \operatorname{rk}(c_+) = \operatorname{rk}(x_0)$. The special case $\lambda = 0$ of Lemma 7 yields $\operatorname{rk}(x_{(10)}) \leqslant \operatorname{rk}(x_{(10)} + (x_{(20)} - u_{(20)}))$. Hence by $(*)$ and $(**)$,

$$\begin{aligned}
\operatorname{rk}(x_0 + x_1) &\leqslant \operatorname{rk}(x_0) + \operatorname{rk}(x_{(10)}) + \operatorname{rk}(u_{(20)})\\
&\leqslant \operatorname{rk}(x_0) + \operatorname{rk}(x_{(10)} + (x_{(20)} - u_{(20)})) + \operatorname{rk}(x_0)\\
&= \operatorname{rk}(x) + \operatorname{rk}(x_0),
\end{aligned}$$

and the proof is complete.

References

1. A. A. Albert, Symmetric and alternate matrices in an arbitrary field. I, *Trans. Amer. Math. Soc.* **43** (1938), 386–436.
2. A. J. Hahn and O. T. OMeara, "The Classical Groups and K-Theory," Springer-Verlag, Berlin, 1989.
3. N. Jacobson, "Structure Theory of Jordan Algebras," The University of Arkansas Lecture Notes in Mathematics, Vol. 5, Fayetteville, 1981.
4. M. Koecher, "Lineare Algebra und analytische Geometrie," Zweite Auflage, Springer-Verlag, 1985.
5. O. Loos, "Jordan Pairs," Lecture Notes in Mathematics, Vol. 460, Springer-Verlag, Berlin, 1975.
6. O. Loos, Finiteness conditions in Jordan pairs, *Math. Z.* **206** (1991), 577–587.
7. O. Loos, On the socle of a Jordan pair, *Collectanea Math.* **40**, No. 2 (1989), 109–125.
8. K. McCrimmon, Reduced elements in Jordan triple systems, *J. Algebra* **97** (1985), 540–564.
9. K. McCrimmon and E. Neher, Coordinatization of triangulated Jordan systems, *J. Algebra* **114** (1988), 411–451.
10. E. Neher, Involutive gradings of Jordan structures, *Comm. Algebra* **9**, No. 6 (1981), 575–599.
11. L. Rowen, "Ring Theory," Vol. I, Academic Press, New York, 1988.

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